Spectral Singularities and a New Method of Generating Tunable Lasers

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Abstract

We use a simple setup based on an infinite planar slab gain medium with no mirrors to explore the possibility of realizing a recently discovered resonance effect related to the mathematical concept of spectral singularity. In particular we determine the range of the gain coefficient g and the width L of the gain region required to achieve this resonance effect. We outline a method that allows for amplifying waves of desired wavelength by adjusting the gain coefficient (pumping intensity). We expect this method to have important practical applications in building tunable lasers.

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1 Introduction

Consider an infinite planar slab gain medium that is aligned along the x-axis, as shown in Figure 1, and a linearly polarized monochromatic electromagnetic (EM) wave traveling along the z-axis: $\vec{E}(z,t) = E\, e^{i(kz-\omega t)}\hat{e}_x$, where E is a constant and $\hat{e}_x$ stands for the unit vector pointing along the positive x-axis. It is easy to show that while traveling through the gain medium the wave is amplified by a factor of e^{gL}, where g is the gain coefficient and L is the width of the gain medium. For a homogeneous and isotropic gain medium that is characterized by a complex refractive index $\mathfrak{n}$, the gain coefficient is related to the wavelength $\lambda := 2\pi c/\omega$ of the incident wave and the imaginary part κ of $\mathfrak{n}$ (also known as the extinction factor) according to [1]

$$g = -\frac{4\pi\kappa}{\lambda}. \tag{1}$$

In practice, one usually places the gain medium between two mirrors (also aligned along the x-axis) to produce a (Fabry-Perot) resonator. In this way one effectively extends the length of the path of the wave through the gain medium and achieves a much larger amplification of the wave for the resonance frequencies. In this article, we will show that it is possible to achieve a similar amplification effect without using any mirrors provided that we choose the width of the gain medium and the value of the gain coefficient in such a way that the system supports a spectral singularity. This is

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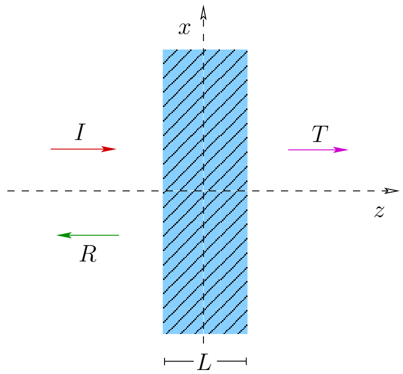

Figure 1: (Color online) Cross section of a planar gain medium (dashed region) in the x-z plane. Arrows labeled by I, R, and T represent the incident, reflected, and transmitted waves.

in effect a method of constructing lasers that as, we will show, allows for adjusting the frequency of the emitted radiation by changing the gain coefficient (pumping intensity.)

Suppose that the gain medium is obtained by doping a host medium of refraction index n_0 and that it is modeled by a two-level atomic system with lower and upper level population densities N_l and N_u, resonance frequency ω_0, and damping coefficient γ. Then the complex permittivity $\varepsilon = \mathfrak{n}^2$ of the system has the form $\varepsilon(z) = \varepsilon_0[1 - \hat{v}(z)]$, where ε_0 is the permittivity of the vacuum,

$$\hat{v}(z) := \left\{ \begin{array}{ccc} \hat{\mathfrak{z}} & \text{for} & |z| < \alpha, \\ 0 & \text{for} & |z| \geq \alpha, \end{array} \right. \qquad \hat{\mathfrak{z}} := 1 - n_0^2 + \frac{\omega_p^2}{\omega^2 - \omega_0^2 + i\gamma\,\omega},$$

$\alpha := L/2$, $\omega_p^2 := (N_l - N_u)e^2/(m\varepsilon_0)$, and e and m are electron's charge and mass, respectively [1, 2].

We can easily construct the following solution of Maxwell's equations for the above system.

$$\vec{E}(z,t) = E\, e^{-i\omega t}\psi(z)\hat{e}_x, \quad \vec{B}(z,t) = -i\omega^{-1}E\, e^{-i\omega t}\psi'(z)\hat{e}_y,$$

where $\hat{e}_y$ is the unit vector along the positive y-axis, and ψ is a continuously differentiable solution of the time-independent Schrödinger equation,

$$-\psi''(z) + v(z)\psi(z) = k^2\psi(z), \tag{2}$$

for the complex barrier potential $v(z) := k^2\hat{v}(z)$.

It turns out that the Schrödinger operator for a complex barrier potential may not have a complete set of eigenfunctions [3]. This is related to the presence of what mathematicians call a spectral singularity [4]. Physically, spectral singularities correspond to the energies k^2 at which both the left and right reflection and transmission coefficients diverge [5, 6]. As a result, they define scattering solutions of (2) that behave exactly like resonances. These states differ from ordinary resonances, because they have real and positive energies. Because for a resonance, the imaginary part of the energy is interpreted as its width, spectral singularities may be viewed as defining resonances with a zero width [5, 7].

For the system considered here, infinite reflection and transmission coefficients mean infinite amplification of an incident EM wave of finite intensity. Obviously, the presence of an intense

radiation field alters the properties of the gain medium and makes our simple model unrealistic. However, as a consequence of this amplification effect and the presence of the background noise, the system emits EM waves of finite intensity at the frequency of the spectral singularity. This is a special lasing effect.

In an optical cavity laser the radiation is confined by a set of mirrors so that it undergoes multiple internal reflections necessary for satisfying the laser threshold condition. In contrast, for the system we consider the amplification effect stems from intrinsic internal reflections that are only present for particular values of the parameters of the system (L and g) and a single frequency associated with these values.

In [5, 3] we consider a possible realization of this phenomenon using two different waveguide systems. Here we explore its consequences for the simpler system of an infinite planar gain medium which is more convenient to analyze. Unlike for the systems considered in [5, 3], here we parameterize the location of spectral singularities using the gain coefficient g which is an easily adjustable quantity in practice. Furthermore, we show that by changing the value of g we can realize the spectral singularity related resonance effect at wavelengths that differ from the resonance wavelength of the gain medium. This may be viewed as a novel method of generating amplified waves of desired wavelength by adjusting the pumping intensity.

In the remainder of this section, we derive a formula that expresses ω_p^2 in terms of g.

Denoting the real part of the complex refractive index $\mathfrak{n}$ by η, so that $\mathfrak{n} = \eta + i\kappa$, and using $\mathfrak{n}^2 = \varepsilon = \varepsilon_0[1 - \hat{v}(z)]$, we have for $|z| < \alpha$:

$$\eta^2 - \kappa^2 = n_0^2 - \frac{\omega_p^2(\omega^2 - \omega_0^2)}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}, \qquad 2\eta\kappa = \frac{\omega_p^2\gamma\,\omega}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}. \tag{3}$$

Setting $\omega = \omega_0$ in (3) and solving for ω_p^2 yields $\omega_p^2 = 2\kappa\gamma\omega_0\sqrt{n_0^2 + \kappa^2}$. Combining this equation with (1) and using $\lambda = 2\pi c/\omega$, we find

$$\omega_p^2 = -\frac{c\gamma\,\omega_0 g}{\omega}\sqrt{n_0^2 + \frac{c^2 g^2}{4\omega^2}}. \tag{4}$$

2 Spectral Singularities of Complex Barrier Potential

The solution of the Schrödinger equation (2) is straightforward. The following is a set of eigenfunctions of the complex barrier potential $v(x)$.

$$\psi_{k,\mathfrak{a}}(x) = \left\{ \begin{array}{ccc} A_-^{\mathfrak{a}} e^{ikx} + B_-^{\mathfrak{a}} e^{-ikx} & \text{for} & x \le -\alpha \\ A_0^{\mathfrak{a}} e^{iwkx} + B_0^{\mathfrak{a}} e^{-iwkx} & \text{for} & |x| < \alpha \\ A_+^{\mathfrak{a}} e^{ikx} + B_+^{\mathfrak{a}} e^{-ikx} & \text{for} & x \ge \alpha \end{array} \right. \tag{5}$$

where $k \in \mathbb{R}^+$, $\mathfrak{a} \in \{1, 2\}$ is a degeneracy label, $A_-^{\mathfrak{a}}$ and $B_-^{\mathfrak{a}}$ are free complex coefficients, $w := \sqrt{1 - \mathfrak{z}}$, and $A_0^{\mathfrak{a}}, B_0^{\mathfrak{a}}, A_+^{\mathfrak{a}}$ and $B_+^{\mathfrak{a}}$ are complex coefficients related to $A_-^{\mathfrak{a}}$ and $B_-^{\mathfrak{a}}$. We can express this relationship most conveniently as $\vec{C}_0^{\mathfrak{a}} = \underline{L}\,\vec{C}_-^{\mathfrak{a}}$ and $\vec{C}_+^{\mathfrak{a}} = \underline{M}\,\vec{C}_-^{\mathfrak{a}}$, where

$$\vec{C}_0^{\mathfrak{a}} := \left(\begin{array}{c} A_0^{\mathfrak{a}} \\ B_0^{\mathfrak{a}} \end{array} \right), \qquad \vec{C}_\pm^{\mathfrak{a}} := \left(\begin{array}{c} A_\pm^{\mathfrak{a}} \\ B_\pm^{\mathfrak{a}} \end{array} \right),$$

$$\underline{L} := \frac{1}{2w} \left(\begin{array}{cc} e^{i\alpha k(w-1)}(w+1) & e^{i\alpha k(w+1)}(w-1) \\ e^{-i\alpha k(w+1)}(w-1) & e^{-i\alpha k(w-1)}(w+1) \end{array} \right),$$

$\underline{M}$ is the transfer matrix:

$$\underline{M} := \frac{1}{4w}\left(\begin{array}{cc} e^{-2i\alpha k}f(w,-\alpha k) & 2i(w^2-1)\sin(2w\alpha k) \\ -2i(w^2-1)\sin(2w\alpha k) & e^{2i\alpha k}f(w,\alpha k)\end{array}\right),$$

and $f(z_1,z_2) := e^{-2iz_1z_2}(1+z_1)^2 - e^{2iz_1z_2}(1-z_1)^2$ for all $z_1, z_2 \in \mathbb{C}$. The spectral singularities are the energy values k^2 for which $M_{22}=0$, i.e., the real k values for which

$$f(w,\alpha k) = 0. \tag{6}$$

In Ref. [3], we have provided a detailed analysis of this equation. Here we give the final result.

First, we introduce the parameters: $\rho := \mathrm{Re}(\hat{\mathfrak{z}}) = \mathrm{Re}(1-w^2)$, $\sigma := \mathrm{Im}(\hat{\mathfrak{z}}) = \mathrm{Im}(1-w^2)$, and the functions:

$$\begin{aligned} g_1(\rho,\sigma) &:= \cos^{-1}\left[\frac{1-\sqrt{(1-\rho)^2+\sigma^2}}{\sqrt{\rho^2+\sigma^2}}\right], & (7) \\ g_2(\rho,\sigma) &:= 1-\rho+\sqrt{(1-\rho)^2+\sigma^2}, & (8) \end{aligned}$$

where $\cos^{-1}$ stands for the principal value of the inverse of cos; in particular, $\cos^{-1}(x) \in [0,\pi]$ for all $x \in [-1,1]$. Then (6) has a solution for a real k if and only if $\sigma > 0$, $\rho < 1$, and for some positive integer n,

$$\sinh^2\left\{\left[\pi n - g_1(\rho,\sigma)\right]\sqrt{1-\frac{2(1-\rho)}{g_2(\rho,\sigma)}}\right\} = \frac{2g_2(\rho,\sigma)}{\rho^2+\sigma^2}. \tag{9}$$

This equation determines an infinite family of curves (labeled by n) in the ρ-σ plane along. See Figure 2. For the ρ and σ values corresponding to a point on one of these curves the spectrum of the complex barrier potential $v(x)$ includes a spectral singularity. Furthermore, the wave number corresponding to this spectral singularity is given by [3]:

$$k = \frac{\pi n - g_1(\rho,\sigma)}{\alpha\sqrt{2g_2(\rho,\sigma)}}. \tag{10}$$

3 Realizing a Spectral Singularity Using a Solid State Gain Medium

For the particular problem we are considering,

$$\rho = 1 - n_0^2 + \frac{\omega_p^2(\omega^2-\omega_0^2)}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}, \qquad \sigma = \frac{-\omega_p^2\gamma\,\omega}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}. \tag{11}$$

In order to gain an insight in the condition of the occurrence of spectral singularities, we first consider the possibility of creating a spectral singularity at the resonance frequency: $\omega = \omega_0$. Then Eqs. (11) simplify considerably, and in view of (4) we find

$$\rho = 1 - n_0^2, \qquad \sigma = \frac{\lambda_0 g}{2\pi}\sqrt{n_0^2 + \left(\frac{\lambda_o g}{4\pi}\right)^2}, \tag{12}$$

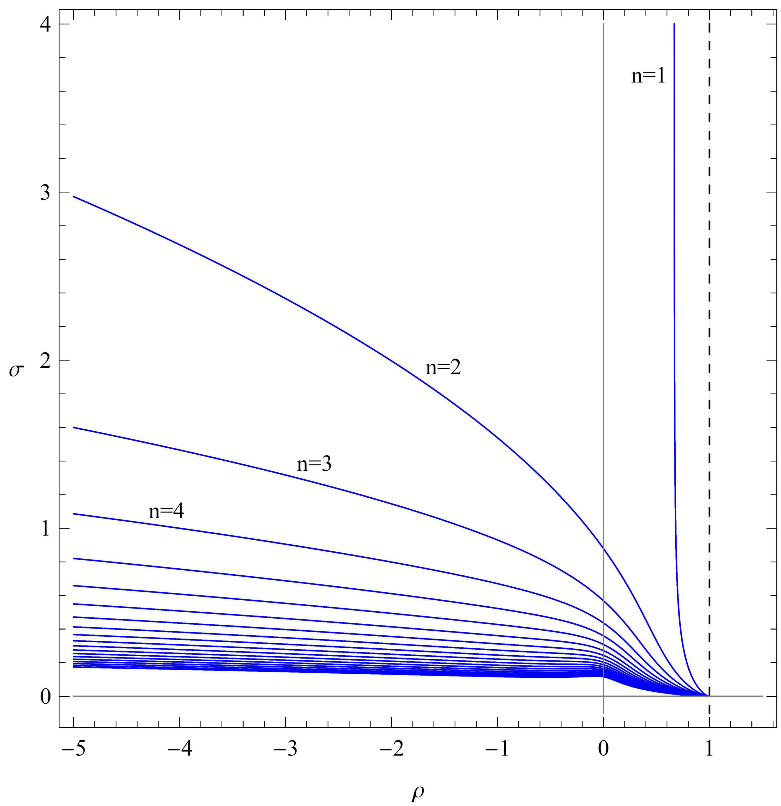

Figure 2: (Color online) Graphs of the curves in the ρ-σ plane along which there is a spectral singularity. The unmarked curves correspond to $n = 5, 6, 7, \cdots, 20$ from top to bottom, respectively. As $n \to \infty$ the curve labeled by n tends to the negative ρ-axis. The dashed line corresponds to $\rho = 1$. The solid gray lines are the coordinate axes.

where $\lambda_0 := 2\pi c/\omega_0$. For a very wide range of practical situations $\lambda_0 g \ll 2\pi$. This together with (12) imply

$$\sigma \approx \frac{\lambda_o g n_0}{2\pi} \ll 1. \tag{13}$$

As seen from Figure 2, this relation suggests that we can tune the parameters of the system to produce a spectral singularities for large values of n. Figure 3 shows the graphs of σ and λ/L as a function of n for the case that $n_0 = 1.76$, $\lambda = \lambda_0$, and $1000 \leq n \leq 10000$. For this range of values of n, both of these functions turn out to have the same general (monotonically decreasing) behavior. This is also true for other typical values of n_0. Using the values $n = 1000, 1500, 2000, 2500, \cdots, 10000$ we could establish that for this range of values of n, both σ and λ/L are approximately inversely proportional to n; $\sigma \approx (4.71 \pm 0.60)/n$ and $\lambda/L \approx (3.19 \pm 0.33)/n$.

Let us now take $\lambda_0 = 800$ nm for the resonance wavelength and again set $n_0 = 1.76$. Then as we increase the value of n from 1000 to 10000, the gain coefficient g (respectively the width of the gain region $L = 2\alpha$) that would correspond to a spectral singularity decreases from 193.98/cm to 25.19/cm (increases from 0.22705 mm to 2.2725 mm). Figure 4 shows the graphs of g and L as a function of n. For $1000 \leq n \leq 10000$, g is a monotonically decreasing function of n while L is monotonically increasing function of n. Using the values of g and L for $n = 1000, 1500, 2000, 2500, \cdots, 10000$, we

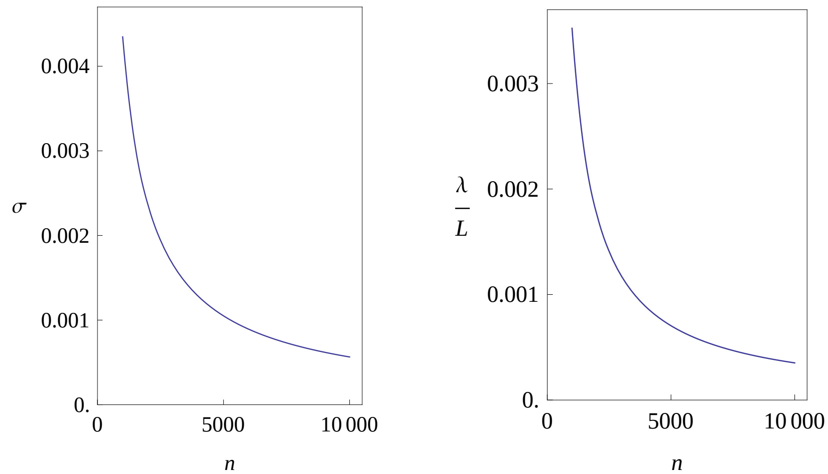

Figure 3: (Color online) Graphs of σ (on the left) and λ/L (on the right) as functions of n for $\lambda = \lambda_0$ and $n_0 = 1.76$.

find that

$$g \approx (2.10 \pm 0.27) \times 10^5/n \ \ \mathrm{cm}^{-1}, \qquad L \approx (2.06 \pm 0.21) \times 10^{-5}\, n \ \ \mathrm{cm}, \qquad g\,L \approx 4.78 \pm 0.61. \tag{14}$$

The range of gain coefficients that we obtain for $1000 \leq n \leq 10000$ is actually quite high for a typical solid state gain media such as a Titanium Sapphire crystal with $n_0 = 1.76$ and $\lambda_0 = 800$ nm, [1]. To obtain a spectral singularity for a typical gain coefficient for this crystal, we need to take a much larger n. This in turn requires a much larger gain region. For example, for $g = 0.2\ \mathrm{cm}^{-1}$, (14) suggests that $n \gtrsim 10^6$ and $L \gtrsim 20$ cm. A direct calculation shows that setting $n = 1.945 \times 10^6$ we find a spectral singularity for $g = 0.200\ \mathrm{cm}^{-1}$ and $L = 44.2$ cm. This is an unrealistically large number for the size of a solid state gain medium with a uniform gain coefficient. Therefore, an experimental study of spectral-singularity-related resonance effect that uses the simple setup we outlined above requires a gain medium with relatively high gain coefficient. Possible alternatives are the gain media used in dye lasers or the semiconductor diode lasers.

4 Realizing a Spectral Singularity Using a Semiconductor Gain Medium

In this section we consider the theoretical plausibility of achieving a spectral singularity related resonance effect using a semiconductor gain medium. Here we disregard the experimental difficulties of realizing our simple setup.

Consider a gain medium used in building a typical diode laser [1] with $n_0 = 3.4$ and

$$\lambda_0 = 0.35 - 24\ \mu\mathrm{m}, \quad \gamma = 1.57 - 6.28 \times 10^{13}\ \mathrm{Hz}, \quad g = (1 - 10) \times 10^4\ \mathrm{m}^{-1}, \quad L = 200 - 500\ \mu\mathrm{m}. \tag{15}$$

To be specific we first examine the case that

$$\lambda_0 = 1.5\ \mu\mathrm{m}, \quad \gamma = 3 \times 10^{13}\ \mathrm{Hz}, \quad g = 5 \times 10^4\ \mathrm{m}^{-1}, \quad L = 350\ \mu\mathrm{m}, \tag{16}$$

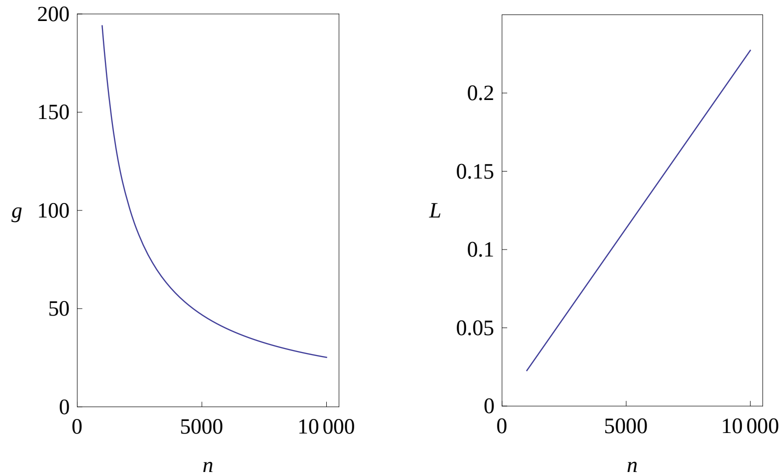

Figure 4: (Color online) Graphs of the gain coefficient g in cm^{-1} (on the left) and the length of the gain region $L = 2\alpha$ in cm (on the right) as functions of n for $\lambda_0 = 800$ nm and $n_0 = 1.76$.

$gL = 17.5$, and $\lambda_0 g = 7.5 \times 10^{-2} \ll 1$. We note that the above value of g is the maximum value at which the diode laser functions. In other words, we are allowed to adjust gain coefficient to any value between 0 and $5 \times 10^4\ \mathrm{m}^{-1}$.

In view of (12) and $\lambda_0 g \ll 1$, we can use (13) to estimate the value of σ. This gives $\sigma \approx 0.04$. We find by inspection that to maintain this value we should take $n \approx 110$. It turns out that for $n = 111$, the necessary gain coefficient is given by $g = 4.9959 \times 10^4\ \mathrm{m}^{-1}$. This in turn corresponds to $L = 24.26\ \mu$m which lies outside the allowed range (15).

Examining the value of the product gL for n between 110 and 114 we see that its average value and standard deviation are respectively 1.21 and 1.8×10^{-7}. Hence at least for this range of values of n, gL is essentially independent of n. Furthermore, g (respectively L) is a decreasing (increasing) function of n. This shows that to obtain realistic values of g and L as given in (15) we should consider larger values for n.

Suppose that we wish to use a sample with $L \approx 350\mu$m. Then given $gL \approx 1.21$, we estimate that $g \approx 3500\mathrm{m}^{-1}$, and in view of (13), $\sigma \approx 0.0028$. This turns out to correspond to $n \approx 1600$. Direct calculation confirms the existence of a spectral singularity with $n = 1588$, $g = 3462.91\ \mathrm{m}^{-1}$, and $L = 350.074\ \mu$m. We have also checked that for $n = 1585 - 1589$, the average and standard deviation of gL are respectively given by 1.21 and 2.88×10^{-9}, i.e., at least for large values of n, gL is essentially independent of n. In particular, $g \approx 1.21/L$.

5 Adjusting the Wavelength of the Emitted Wave by Changing the Pumping Intensity

Consider the following practically important scenario. Suppose that we fix the width of the gain region L so that there is a spectral singularity at the resonance wavelength λ_0 for a particular value of the gain coefficient $g_\star$ and the label n that we denote by $n_\star$. If we vary the value of the gain coefficient g in the vicinity of $g_\star$, we can adjust it to create a new spectral singularity for some λ and

g (m^{-1})	λ (μm)	n	g (m^{-1})	λ (μm)	n
60929	1.4305	1668	65440	1.4280	1671
60966	1.5768	1512	65475	1.5710	1509
62414	1.4297	1669	66982	1.4271	1672
62450	1.5779	1511	67015	1.5810	1508
63918	1.4288	1670	68542	1.4263	1673
63953	1.5789	1510	68575	1.5820	1507

Table 1: The spectral singularities for a semiconductor gain medium with $n_0 = 3.4$, $\lambda_0 = 1.5$ μm, $\gamma = 3 \times 10^{13}$ Hz, $L = 350.1$ μm, obtained at various wavelengths as one changes the gain coefficient g in the range 60000-70000 m^{-1}.

n in the vicinity of λ_0 and $n_\star$. This is an important observation, for we can use it as a method of generating amplified waves of wavelength close to λ_0 by adjusting the value of the gain coefficient, i.e., carefully tuning the pumping intensity. As far as we know, this is the first instance of a method that uses the change in pumping intensity to fine tune the wavelength of the emitted wave. This possibility marks an important distinction between the spectral singularity related resonances and the usual resonances that one encounters in the usual optical resonators.

The mathematical implementation of this idea involves the following steps. First, we express (9) and (10) as equations in the three variables n, g, and λ. Next, we use (10) to express n in terms of g and λ. We can summarize the result as $n = \mathscr{F}_n(g, \lambda)$. Finally, we substitute this expression in (9) to obtain an equation of the form $\mathscr{F}(g, \lambda) = 0$. The spectral singularities are located on the curve C defined by this equation in the g-λ plane, but not all the points on this curve correspond to a spectral singularity. This is because by eliminating the variable n in our calculations we have ignored the fact that n takes integer values only. We can graphically impose this condition by plotting the curves C_n defined by $\mathscr{F}_n(g, \lambda) = n$ for each n in the vicinity of $n_\star$. The intersection of C with C_n gives the points in the g-λ plane that correspond to spectral singularities.

As a concrete example consider a diode laser gain medium with

$$n_0 = 3.4, \quad \lambda_0 = 1.5\ \mu\text{m}, \quad \gamma = 3 \times 10^{13}\ \text{Hz}, \quad g_\star = 3463\ \text{m}^{-1}, \qquad L = 350.1\ \mu\text{m}, \qquad (17)$$

that corresponds to setting $n_\star = 1588$. Figure 5 shows the graphs of the curves C and C_n with $n = 1500, 1520, 1540, \cdots, 1700$ and $n = 1588$. For this range of values of n the gain coefficient is within the range given in (15). As seen from this figure to amplify a wave of wavelength $\lambda \neq \lambda_\star = 1.5$ μm, we need to increase the gain coefficient. For a particular discrete set of values of $g > g_\star$ we obtain spectral singularities at certain wavelengths grouped in pairs with very small difference and ranging between 1.4 and 1.6 μm. Table 1 gives the wavelength of the spectral singularities obtained by changing g in the range 60000-70000 m^{-1}. According to this table, this allows for amplifying waves with wavelength in the ranges 1.4263-1.43051 μm and 1.5768-1.5820 μm.

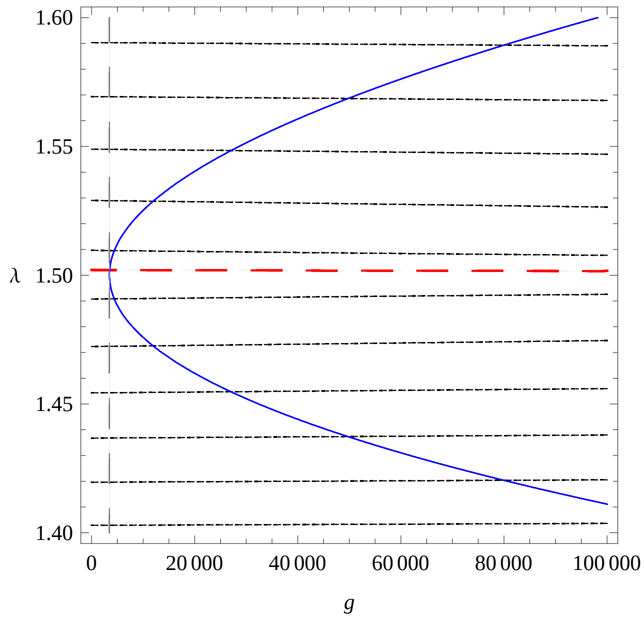

Figure 5: (Color online) Graphs of the curves C (solid, blue curve) and C_n (the dotted black curves) for $n = 1500, 1520, 1540, \cdots, 1700$ (from top to bottom) in the g-λ plain. g and λ values are given in units of μm and m^{-1}, respectively. The vertical dashed (gray) line is the plot of $g = g_\star = 3463$ m^{-1}. The thick dashed red curve is the graph of C_{1588}. The spectral singularities appear in the intersection points of C and C_n.

6 Concluding Remarks

In this article, we examined the possibility of realizing the spectral-singularity-related resonance (lasing) effect by using the gain coefficient to parameterize the problem. We showed that an experimental verification of this effect that involves a typical solid state gain medium is unrealistic and that we needed a gain medium of relatively high gain coefficient. We therefore explored the possibility of using a semiconductor gain medium.

More importantly, we proposed a method that allows for amplifying waves of wavelength different from the resonance wavelength of the gain medium by simply adjusting the gain coefficient. This is an interesting observation as one can change the gain coefficient by changing the pumping intensity. The toy model we used to examine the above issues is probably too simple for modeling a real experiment. Here we considered this system because it revealed some of the remarkable consequences of the spectral singularity-related resonance effect.

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